On the path to minimal-length descriptions, guided by Kolia and Sasha

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Some good books.

This talk in one slide

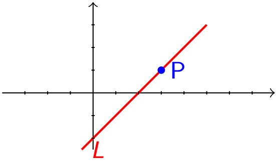
Common wisdom: Kolmogorov complexity is about optimal compression/decompression, where compression is not effective, but decompression is.

> *"... in the framework of Kolmogorov complexity we have no compression algorithm and deal only with decompression algorithms."*
>
> — Alexander Shen, Around Kolmogorov complexity: basic notions and results, 2015.

We shall see natural circumstances where compression to close to minimum description length is not only effective but actually **efficient** (and decompression is effective but not efficient).

A preparatory puzzle

- Kolia and Sasha want to agree on a secret key.
- Problem is that we hear everything they say.
- Kolia knows line $L : y = a_1 x + a_0$;
  Sasha knows point P: (b_1, b_2);
- L : $2n$ bits of information (intercept, slope in $\mathbb{F}_{2^n}$).
- P: $2n$ bits of information (the 2 coord. in $\mathbb{F}_{2^n}$).
- Total information in $(L, P) = 3n$ bits; mutual information of L and $P = n$ bits.



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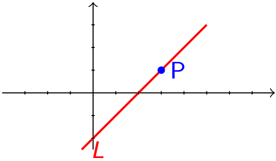
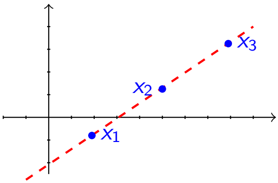


SOLUTION:

- Kolia tells a_1 to Sasha.
- Sasha, knowing that $P \in L$, finds L.
- Kolia and Sasha use a_0 as a secret key.
- It works! We have heard a_1, but a_1 and a_0 are independent.

The real puzzle



Kolia:	x_1
Sasha:	x_2
Andrei:	x_3

Points x_1, x_2, x_3 belong to one line in the affine plane over $\mathbb{F}_{2^n}$

Each point has $2n$ points of information, but together they have $5n$ bits of information.

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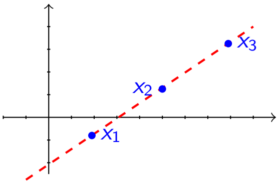

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QUESTION: Can they agree on a secret key by discussing in this room, where we all hear what they say?

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We fix some optimal $\mathcal{U}$ once and forever.

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$C(x) \leq \log n + O(1)$

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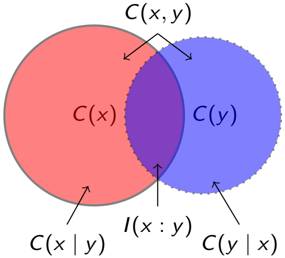
- Mutual information of x and y :
  $I(x : y) := C(x) - C(x \mid y)$.



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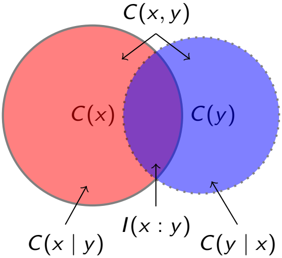
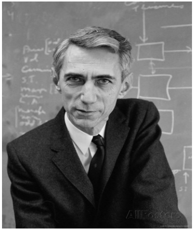
- **Chain Rule** [Kolmogorov, Levin] $C(x, y) =^{+} C(x) + C(y \mid x)$

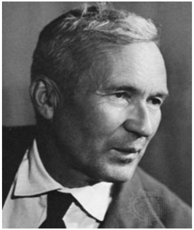
  where the notation $=^{+}$ hides $\pm O(\log n)$

Corollary. $I(x : y) =^{+} C(x) + C(y) - C(x, y) =^{+} I(y : x)$



IT vs. AIT (or Shannon vs. Kolmogorov)





The word **random** is used in computer science in two ways:

(1) **random** process: a process whose outcome is uncertain, e.g. a series of coin tosses.

(2) **random** object: something that lacks regularities, patterns, is incompressible.

Information Theory (IT) focuses on (1).

Algorithmic Information Theory (AIT, also known as Kolmogorov complexity) focuses on (2).

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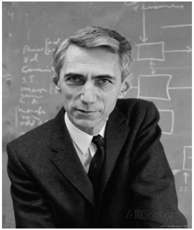

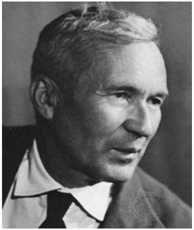

IT (à la Shannon)

- Data is the realization of a random variable X.
- The model: a stochastic process generates the data.
- Amount of information in the data:

$H(X) = \sum p_i \log(1/p_i)$ (Shannon entropy).

AIT (Kolmogorov complexity)

- Data is just an individual string x
- There is no generative model.
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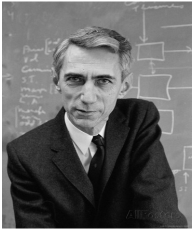

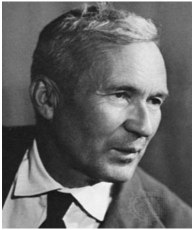

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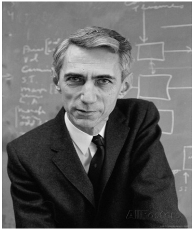

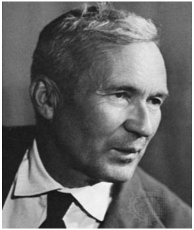

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Short programs and communication protocols

Alice has x Bob has y.

They run an interactive protocol.

$$\begin{array}{c} \xrightarrow{\quad t_1 \quad} \\ \xleftarrow{\quad t_2 \quad} \\ \xrightarrow{\quad t_3 \quad} \\ \vdots \\ \xrightarrow{\quad t_k \quad} \end{array}$$

Bob has x

QUESTION: What is the communication complexity?

Can it be $C(x \mid y)$? Is there a protocol that comes close to this?

Scenario: Alice and Bob are computationally unbounded

Alice has x, Bob has y. They run a protocol. At the end, Bob has x.

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- (Vereshchagin, 2014) The randomized communication complexity of computing $C(x \mid y)$ with precision ϵn is $0.99n$.

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- (Zimand, 2014) The size of the list in Teutsch's result is $O(n^{6+\epsilon})$.

Dagstuhl 2003

Scenario: Alice is algorithmically bounded and holds advice information

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- (Musatov, Romashchenko, Shen, 2009) **Space-bounded version of Muchnik's Th.**:

  For every space bound s, Alice on x and some $O(\log^3 n)$-long advice can compute in **polynomial space** a program p for x given y with space complexity $O(s) + \mathrm{poly}(n)$ and $|p| = C^{\mathrm{space}=s}(x \mid y) + O(\log n)$.

Scenario: Alice is algorithmically bounded and knows $C(x \mid y)$.

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- If we drop the poly time requirement, the overhead can be reduced to $O(\log n)$.
- The overhead cannot be less than $\log n - \log\log n - O(1)$, for total computable compressors.

Scenario: Alice is algorithmically bounded and knows an upper bound of $C(x \mid y)$.

Alice has x, Bob has y. Alice wants a program for x given y (which she can send to Bob, to communicate x).

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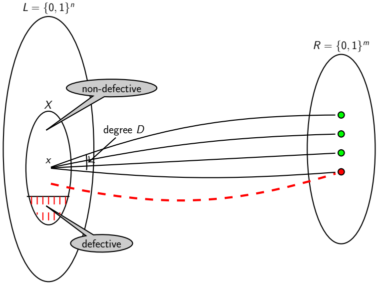
- Alice on input x and k can find a program p for x with $|p| \leq m$ by exhaustive search.
- (Zimand, 2017) (Bauwens, Zimand, 2019) Alice on input (x, m) can compute in **probabilistic polynomial time** a program for x given y of length $m + O(\log^2(n/\epsilon))$, with probability error ϵ.

Theorem (Bauwens, Zimand, 2019)

(Bauwens, Zimand, 2019) Alice on input (x, m), where $m \geq C(x \mid y)$, can compute in **probabilistic polynomial time** *a program for x given y of length $m + O(\log^2(n/\epsilon))$, with probability error ϵ.*



- $f : L \times [D] \to R$, used for fingerprinting.
- $f(x, 1), \ldots, f(x, D)$ are the fingerprints of x.
- X is the list of candidates, we want to identify which candidate is x.
- A fingerprint is heavy for X, if it has more $2D$ pre-images in X.
- x is ϵ-defective for X if it has more than ϵD heavy fingerprints.

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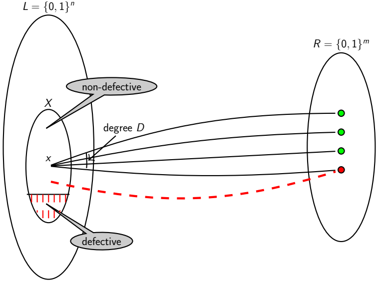

- $f : \{0,1\}^n \times [D] \to \{0,1\}^m$ is a $k \to_\epsilon k$ condenser , if for every r.v. X with min - entropy k, $f(X, U_D)$ is ϵ-close to having min-entropy k.

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$\Pr[X = x] \leq 2^{-k}$ for all x

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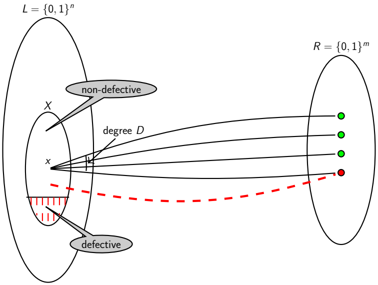

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Buhrman, Fortnow, Laplante 2001
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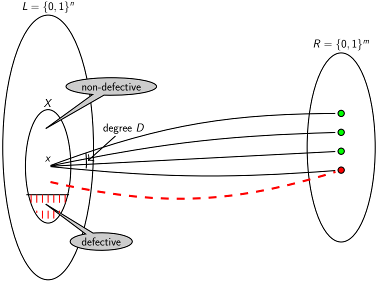

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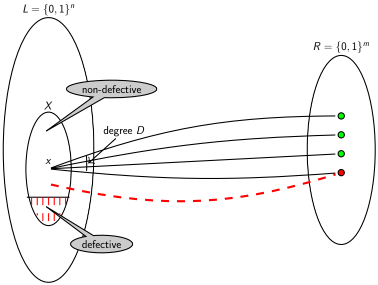

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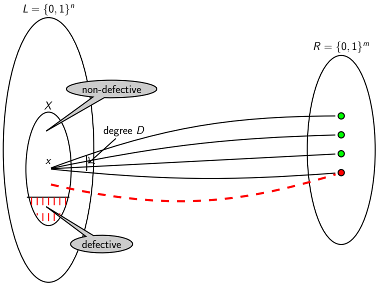

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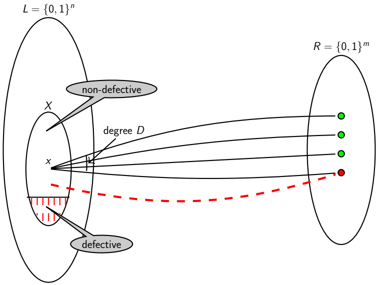

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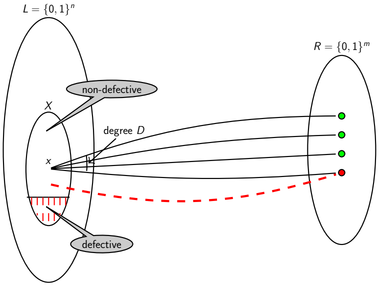

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building on the Guruswami-Umans-Vadhan extractor

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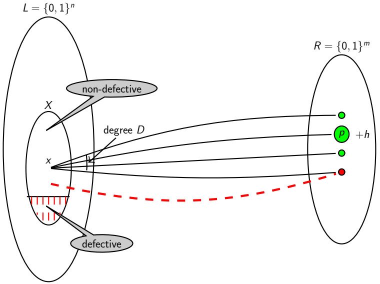

- x has complexity $C(x) \leq m$.
- $f : \{0,1\}^n \times [D] \to \{0,1\}^m$, the explicit ϵ-conductor.
- x has complexity $C(x) \leq m$.
- $f(x, 1), \ldots, f(x, D)$ are the fingerprints of x.
- **Compress x:** pick randomly p, one of the fingerprints. Append h, a short hash-code of x. Output (p, h). Length: $m + |h|$.
- **Decompression:** we want to reconstruct x from (p, h).
- X the set of strings with complexity $\leq m$ (list of candidates). we want to identify which candidate is x.

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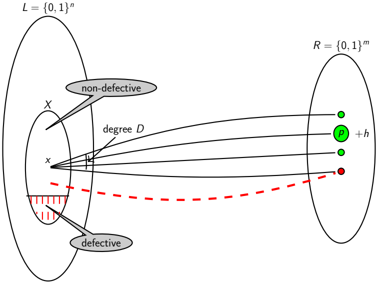

- Decompression: we want to reconstruct x from (p, h).
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- Case 1: x is non-defective. With prob $1 - \epsilon$, we reduce the list of candidates to the $2D$-preimages of p.
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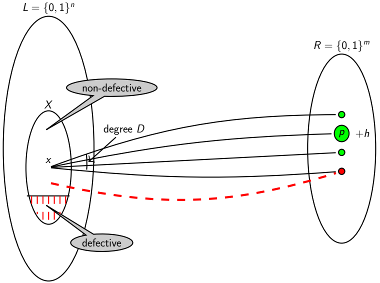

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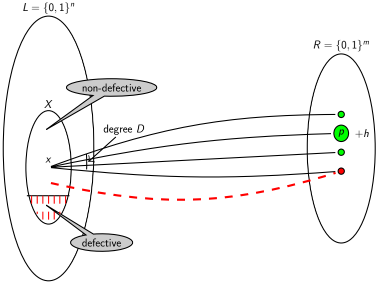

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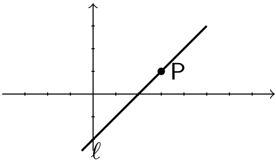
- Case 2: x is defective. We reduce the list of candidates to the set of defective elements, so we reduce the list by $1/2$.
- Problem: We do not know which of Case 1 or Case 2 is true.
- We collect the candidates as if Case 1 is true, so we keep only the first $2D$ preimages of p. Then reduce as in Case 2.
- At the end we have collected $m \times 2D$ candidates.
- We identify x using h, the short hash code.

Distributed compression: a simple example

- Alice knows a line ℓ; Bob knows a point $P \in \ell$; They want to send ℓ and P to Zack.
- ℓ : $2n$ bits of information (intercept, slope in GF[2^n]).
- P : $2n$ bits of information (the 2 coord. in GF[2^n]).
- Total information in $(\ell, P) = 3n$ bits; mutual information of ℓ and $P = n$ bits.
- If Alice and Bob get together, they need to send $3n$ bits. What if they compress separately?



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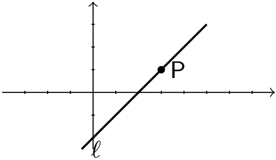

QUESTION 1:

Alice can send $2n$ bits, and Bob n bits. Is the geometric correlation between ℓ and P crucial for these compression lengths?

Ans: No. Same is true (modulo a $\mathrm{polylog}(n)$ overhead.) if Alice and Bob each have $2n$ bits of information, with mutual information n, in the sense of Kolmogorov complexity.

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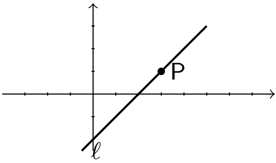


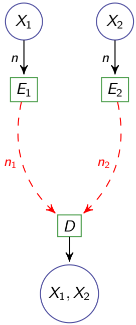
QUESTION 2:

Can Alice send $1.5n$ bits, and Bob $1.5n$ bits? Can Alice send $1.74n$ bits, and Bob $1.26n$ bits?

Ans: Yes and Yes (modulo a $\mathrm{polylog}(n)$ overhead.)

Distributed compression (IT view): Slepian-Wolf Theorem

- The classic Slepian-Wolf Th. is the analog of Shannon Source Coding Th. for the distributed compression of **memoryless** sources.
- Memoryless source: (X_1, X_2) consists of n independent draws from a joint distribution $p(b_1, b_2)$ on pair of bits.
- Encoding: $E_1 : \{0,1\}^n \to \{0,1\}^{n_1}$, $E_2 : \{0,1\}^n \to \{0,1\}^{n_2}$.
- Decoding: $D : \{0,1\}^{n_1} \times \{0,1\}^{n_2} \to \{0,1\}^n \times \{0,1\}^n$.
- Goal: $D(E_1(X_1), E_2(X_2)) = (X_1, X_2)$ with probability $1 - \epsilon$.
- It is necessary that $n_1 + n_2 \geq H(X_1, X_2) - \epsilon n$, $n_1 \geq H(X_1 \mid X_2) - \epsilon n$, $n_2 \geq H(x_2 \mid x_1) - \epsilon n$.



Theorem (Slepian, Wolf, 1973)

There exist encoding/decoding functions E_1, E_2 and D satisfying the goal for all n_1, n_2 satisfying

$$n_1 + n_2 \geq H(X_1, X_2) + \epsilon n,\ n_1 \geq H(X_1 \mid X_2) + \epsilon n,\ n_2 \geq H(X_2 \mid X_1) + \epsilon n.$$

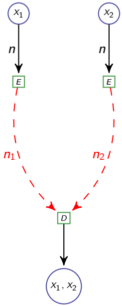
It holds for any constant number of sources.

Slepian-Wolf Th.: Some comments

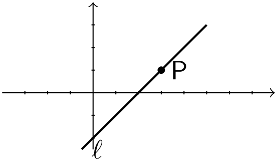
Theorem (Slepian, Wolf, 1973)

There exist encoding/decoding functions E_1, E_2 *and* D *such that* $n_1 + n_2 \geq H(X_1, X_2) + \epsilon n$, $n_1 \geq H(X_1 \mid X_2) + \epsilon n$, $n_2 \geq H(X_2 \mid X_1) + \epsilon n$.

- Even if (X_1, X_2) are compressed together, the sender still needs to send $\approx H(X_1, X_2)$ many bits.
- **Strength of S.-W. Th. :** distributed compression = centralized compression, for memoryless sources.
- **Shortcoming of S.-W. Th. :** Memoryless sources are very simple. The theorem has been extended to stationary and ergodic sources (Cover, 1975), which are still pretty lame.



- Recall: Alice knows a line ℓ; Bob knows a point $P \in \ell$; They want to send ℓ and P to Zack.
- There is no generative model.
- Correlation can be described with the complexity profile: $C(\ell) = 2n, C(P) = 2n, C(\ell, P) = 3n$.
- Is it possible to have distributed compression based only on the complexity profile?
- If yes, what are the possible compression lengths?



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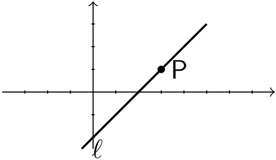


Necessary conditions: Suppose we want encoding/decoding procedures so that $D(E_1(x_1), E_2(x_2)) = (x_1, x_2)$ with probability $1 - \epsilon$, for all strings x_1, x_2. Then, for infinitely many x_1, x_2,

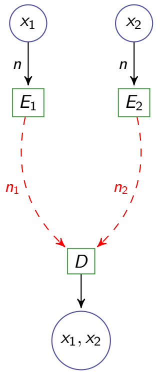
$$
\begin{aligned}
|E_1(x_1)| + |E_2(x_2)| &\geq C(x_1, x_2) + \log(1 - \epsilon) - O(1) \\
|E_1(x_1)| &\geq C(x_1 \mid x_2) + \log(1 - \epsilon) - O(1) \\
|E_2(x_2)| &\geq C(x_2 \mid x_1) + \log(1 - \epsilon) - O(1)
\end{aligned}
$$

Kolmogorov complexity version of the Slepian-Wolf Theorem

Theorem ((Z. 2017), (Bauwens, Z. 2019))

There exist probabilistic poly.-time algorithms E and algorithm D such that for all integers n_1, n_2 and n-bit strings x_1, x_2,
if $n_1 + n_2 \geq C(x_1, x_2)$, $n_1 \geq C(x_1 \mid x_2)$,
$n_2 \geq C(x_2 \mid x_1)$,
then

- *E on input (x_i, n_i) outputs a string p_i of length $n_i + O(\log^2 n)$, for $i = 1, 2$,*
- *D on input (p_1, p_2) outputs (x_1, x_2) with probability 0.99.*



There is an analogous version for any constant number of sources.

Proof sketch (1/2)

- Alice has x_1 and n_1.

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- n_1, n_2 satisfy the Slepian-Wolf constraints:

  $n_1+n_2 \geq C(x_1, x_2), n_1 \geq C(x_1 \mid x_2), n_2 \geq C(x_2 \geq x_1)$.

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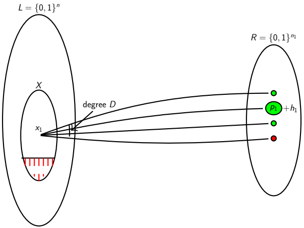

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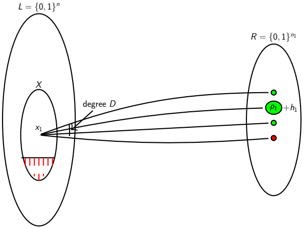

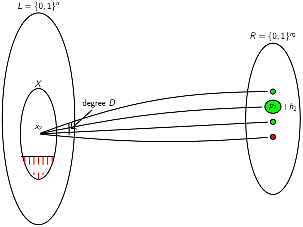

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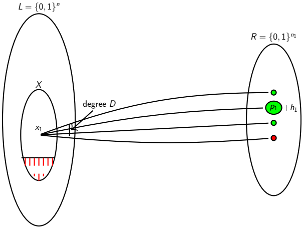

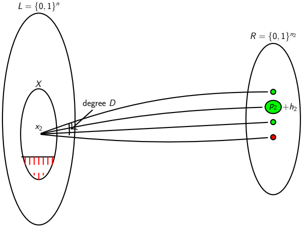

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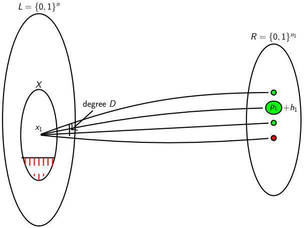
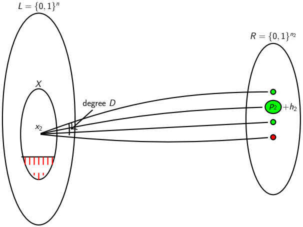
- Alice compresses x_1 by choosing a random neighbor p_1 + short hash-code h_1.
- Bob compresses x_2 by choosing a random neighbor p_2 + short hash-code h_2.



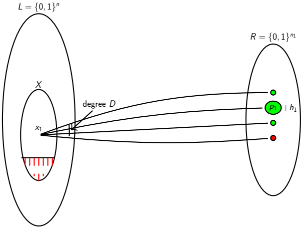


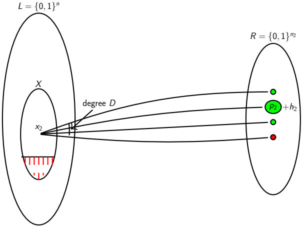
Proof-sketch (2/2)

- How to reconstruct (x_1, x_2) from (p_1, h_1) and (p_2, h_2)
- Enumerate the initial list of candidates: all pairs x_1', x_2' with

  $$n_1 + n_2 \geq C(x_1', x_2'), n_1 \geq C(x_1' \mid x_2'), n_2 \geq C(x_2' \geq x_1').$$

- Apply a cascade of two filters to each enumerated pair.
- Pair $(x_1', *)$ passes the first filter if (p_1, h_1) is the compressed code of x_1'.
- Pair $(*, x_2')$ passes the second filter if (p_2, h_2) is the compressed code of x_2'.
- With high probability, only (x_1, x_2) survive the two filters.





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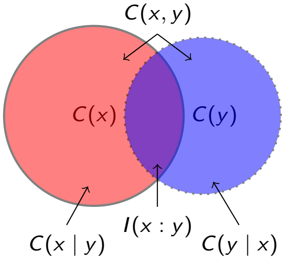
- The $O(\log^2 n)$ overhead can be reduced to $O(\log n)$, but compression is no longer in polynomial time.

Operational characterization of mutual information



$C(x)$ = length of a shortest description of x.
$C(x \mid y)$ = length of a shortest description of x given y.

$\vdots$

Mutual information of x and y is defined by a formula:
$I(x:y) = C(x) + C(y) - C(x,y)$.
Also, $I(x:y) =^+ C(x) - C(x \mid y)$,
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All the regions except the center have an operational meaning.

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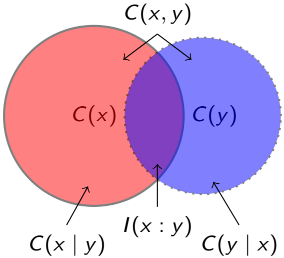

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Does $I(x:y)$ have an operational meaning?

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- This was known in the setting of Information Theory (Shannon entropy, etc.) for memoryless and stationary ergodic sources.
- (Romashchenko, Z., 2018) Characterization holds in the framework of Kolmogorov complexity.

Secret key agreement protocol:

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Theorem (Characterization of the mutual information)

1. *There is a protocol that for every n-bit strings x and y allows to compute with high probability a shared secret key of length $I(x:y)$ (up to $-O(\log n)$).*
2. *No protocol can produce a longer shared secret key (up to $+O(\log n)$).*

Characterization of mutual information: the positive part

Theorem

There exists a secret key agreement protocol with the following property: if

- *Alice knows* x, ϵ, *and the complexity profile of* (x, y),
- *Bob knows* y, ϵ, *and the complexity profile of* (x, y),

then with probability $1 - \epsilon$ *they obtain a string* z *such that,*

$|z| \geq I(x : y) - O(\log(n/\epsilon))$

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and $C(z \mid \text{transcript}) \geq |z| - O(\log(1/\epsilon))$. /* no information leakage */

Secret key agreement: sketch of the general protocol

- Alice and Bob want to agree on a secret key.
- they can only communicate through a public channel.
- Alice knows x; Bob knows y;
- $C(x \mid y) =^{+} n_1$
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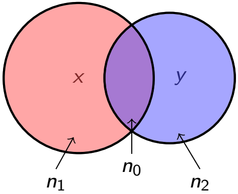
- $I(x : y) =^{+} n_0$.



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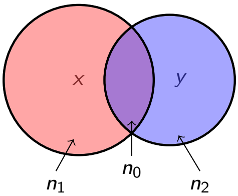

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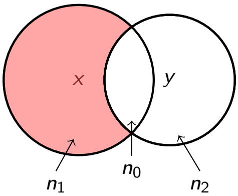

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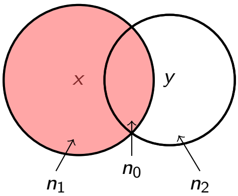

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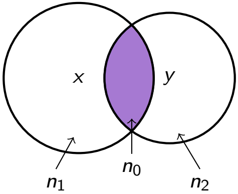

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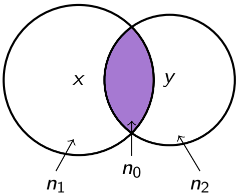

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- Adversary gets p but learns nothing about z.

Characterization of mutual information: the negative part.

Theorem

Let x and y be input strings of length n on which the protocol succeeds with error probability ϵ so that with prob $1-\epsilon$ Alice and Bob have at the end the same z, and $C(z \mid t) \geq |z| - \delta(n)$.

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Kaced-Romashchenko-Vereshchagin 2017
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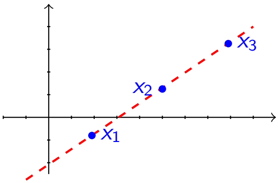

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points x_1, x_2, x_3 belong to one line in the affine plane over $\mathbb{F}_{2^n}$

Each point has $2n$ points of information, but together they have $5n$ bits of information.

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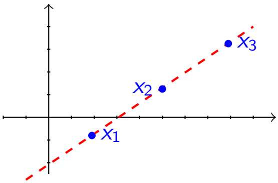

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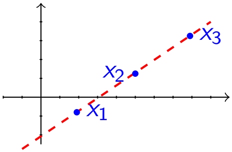

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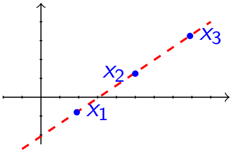

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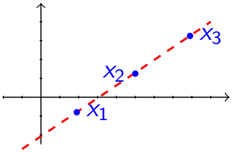

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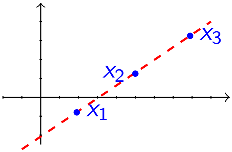

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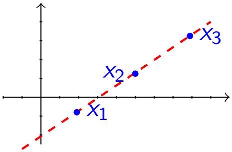

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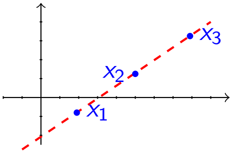

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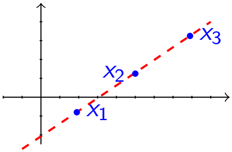
- $n_1, n_2, n_3 = 1.5n$ satisfy the requirements. By S.-W. Th., they can each send $1.5n$ bits.
- We have heard $4.5n$ bits, but they have $5n$ bits.
- They compress their $5n$ bits conditional to our $4.5n$ bits, and obtain $0.5n$ secret bits.

The puzzle: Solution



Kolia: x_1
Sasha: x_2
Andrei: x_3

Points x_1, x_2, x_3 belong to one line in the affine plane over $\mathbb{F}_{2^n}$

Each point has $2n$ points of information, but together they have $5n$ bits of information.

QUESTION: Can they agree on a secret key by discussing in this room, where we all hear what they say?

- Kolia, Sasha, Andrei send, respectively, n_1, n_2, n_3 bits, so that at the end they all know all points.
- Information requirements: $n_i \geq n$, $n_i + n_j \geq 3n$, for all $i, j \in \{1, 2, 3\}$.
- $n_1, n_2, n_3 = 1.5n$ satisfy the requirements. By S.-W. Th., they can each send $1.5n$ bits.
- We have heard $4.5n$ bits, but they have $5n$ bits.
- They compress their $5n$ bits conditional to our $4.5n$ bits, and obtain $0.5n$ secret bits.
- (Romashchenko, Z. 2018) This is the best they can do, they cannot obtain a longer secret key.

References

- M. Zimand, Kolmogorov complexity version of Slepian-Wolf coding, STOC 2017, available on arxiv `https://arxiv.org/abs/1511.03602`
- A. Romashchenko and M. Zimand, An operational characterization of mutual information in algorithmic information theory, ICALP 2018, available at ECCC `https://eccc.weizmann.ac.il/report/2018/043`
- B. Bauwens and M. Zimand, Almost minimum-description length compression and Slepian-Wolf coding in probabilistic polynomial time, 2019, in preparation.

Happy Birthday, Kolia and Sasha

https://www.youtube.com/watch?v=lKIGZsuHQ4U&t=201s