Kolmogorov complexity version of Slepian-Wolf coding

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This work in a sentence

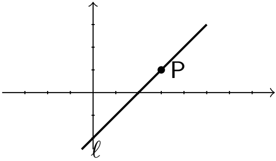
When we compress correlated pieces of data,

Distributed Compression = **Centralized** Compression

and this is true even for a very general definition of correlation based on Kolmogorov complexity.

Distributed compression: a simple example

- Alice knows a line ℓ; Bob knows a point $P \in \ell$; They want to send ℓ and P to Zack.
- ℓ : $2n$ bits of information (intercept, slope in $GF[2^n]$).
- P : $2n$ bits of information (the 2 coord. in $GF[2^n]$).
- Total information in $(\ell, P) = 3n$ bits; mutual information of ℓ and $P = n$ bits.
- If Alice and Bob get together, they need to send $3n$ bits. What if they compress separately?



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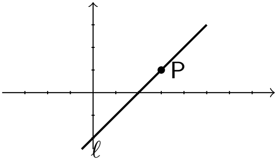

QUESTION 1:

Alice can send $2n$ bits, and Bob n bits. Is the geometric correlation between ℓ and P crucial for these compression lengths?

Ans: No. Same is true (modulo a $\mathrm{polylog}(n)$ overhead.) if Alice and Bob each have $2n$ bits of information, with mutual information n, in the sense of Kolmogorov complexity.

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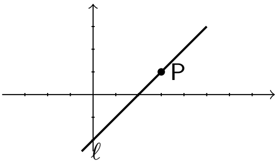

QUESTION 2:

Can Alice send $1.5n$ bits, and Bob $1.5n$ bits? Can Alice send $1.74n$ bits, and Bob $1.26n$ bits?

Ans: Yes and Yes (modulo a $\mathrm{polylog}(n)$ overhead.)

IT vs. AIT

IT (à la Shannon)

• Data is the realization of a random variable X.
• The model: a stochastic process generates the data.
• Amount of information in the data: $H(X)$ (Shannon entropy).

AIT (Kolmogorov complexity)

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Kolmogorov complexity

Fix U a universal Turing machine.

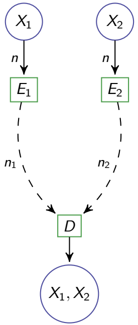
p is a description of x if $U(p) = x$. p is a description of x given y if $U(p, y) = x$.

$C(x) = \min\{|p| \mid p \text{ is a description of } x.\}$

$C(x \mid y) = \min\{|p| \mid p \text{ is a description of } x \text{ given } y.\}$

Distributed compression (IT view): Slepian-Wolf Theorem

- The classic Slepian-Wolf Th. is the analog of Shannon Source Coding Th. for the distributed compression of **memoryless** sources.
- Memoryless source: (X_1, X_2) consists of n independent draws from a joint distribution $p(b_1, b_2)$ on pair of bits.
- Encoding: $E_1 : \{0,1\}^n \to \{0,1\}^{n_1}$, $E_2 : \{0,1\}^n \to \{0,1\}^{n_2}$.
- Decoding: $D : \{0,1\}^{n_1} \times \{0,1\}^{n_2} \to \{0,1\}^n \times \{0,1\}^n$.
- Goal: $D(E_1(X_1), E_2(X_2)) = (X_1, X_2)$ with probability $1 - \epsilon$.
- It is necessary that $n_1 + n_2 \geq H(X_1, X_2) - \epsilon n$, $n_1 \geq H(X_1 \mid X_2) - \epsilon n$, $n_2 \geq H(x_2 \mid x_1) - \epsilon n$.



Theorem (Slepian, Wolf, 1973)

There exist encoding/decoding functions E_1, E_2 and D satisfying the goal such that

$$n_1 + n_2 \geq H(X_1, X_2) + \epsilon n,\ n_1 \geq H(X_1 \mid X_2) + \epsilon n,\ n_2 \geq H(X_2 \mid X_1) + \epsilon n.$$

It holds for any constant number of sources.

Slepian-Wolf Th.: Some comments

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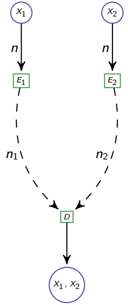
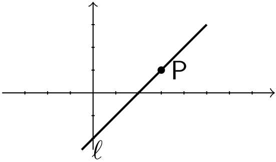
- Even if (X_1, X_2) are compressed together, the sender still needs to send $\approx H(X_1, X_2)$ many bits.
- **Strength of S.-W. Th. :** distributed compression = centralized compression, for memoryless sources.
- **Shortcoming of S.-W. Th. :** Memoryless sources are very simple. The theorem has been extended to stationary and ergodic sources (Cover, 1975), which are still pretty lame.



- Recall: Alice knows a line ℓ; Bob knows a point $P \in \ell$; They want to send ℓ and P to Zack.
- There is no generative model.
- Correlation can be described with the complexity profile: $C(\ell) = 2n, C(P) = 2n, C(\ell, P) = 3n$.
- Is it possible to have distributed compression based only on the complexity profile?
- If yes, what are the possible compression lengths?



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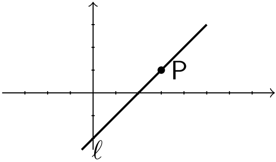


Necessary conditions: Suppose we want encoding/decoding procedures so that $D(E_1(x_1), E_2(x_2)) = (x_1, x_2)$ with probability $1 - \epsilon$, for all strings x_1, x_2.
Then, for infinitely many x_1, x_2,

$$\begin{aligned} |E_1(x_1)| + |E_2(x_2)| &\geq C(x_1, x_2) + \log(1 - \epsilon) - O(1) \\ |E_1(x_1)| &\geq C(x_1 \mid x_2) + \log(1 - \epsilon) - O(1) \\ |E_2(x_2)| &\geq C(x_2 \mid x_1) + \log(1 - \epsilon) - O(1) \end{aligned}$$

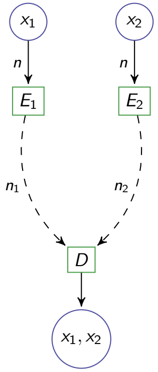
MAIN RESULT: Kolmogorov complexity version of the Slepian-Wolf Theorem

Theorem

There exist probabilistic poly.-time algorithms E_1, E_2 and algorithm D such that for all integers n_1, n_2 and n-bit strings x_1, x_2,
if $n_1 + n_2 \geq C(x_1, x_2)$, $n_1 \geq C(x_1 \mid x_2)$, $n_2 \geq C(x_2 \mid x_1)$,

then

- *E_i on input (x_i, n_i) outputs a string p_i of length $n_i + O(\log^3 n)$, for $i = 1, 2$,*
- *D on input (p_1, p_2) outputs (x_1, x_2) with probability $1 - 1/n$.*



There is an analogous version for any constant number of sources.

Some comments

- Compression takes polynomial time. Decompression is slower than any computable function. This is unavoidable at this level of optimality (compression at close to minimum description length).

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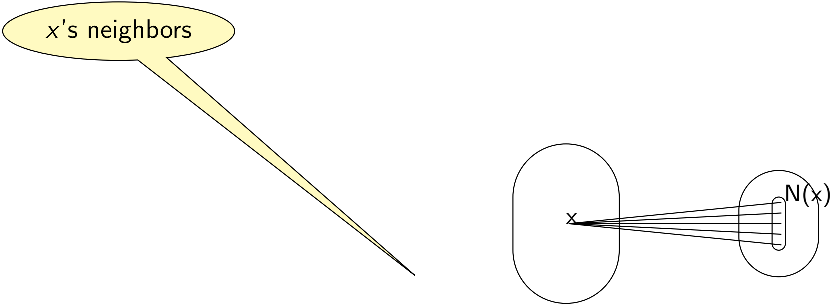
- The $O(\log^3 n)$ overhead can be reduced to $O(\log n)$, but compression is no longer in polynomial time.

Proof sketch

Graphs with the rich owner property

Bipartite graph G, with left degree D;
parameters k, δ;



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small regime case: $|B| \leq 2^k$
x owns $(1-\delta)$ of $N(x)$

large regime case: $|B| > 2^k$
at least fraction $(1-\delta)$ of $y \in N(x)$ have $\deg_B(y) \leq (2/\delta^2)|B|D/2^k$
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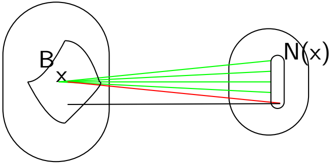

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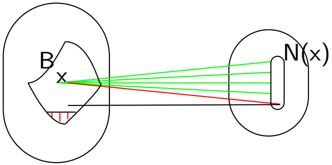

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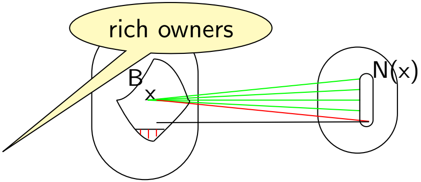

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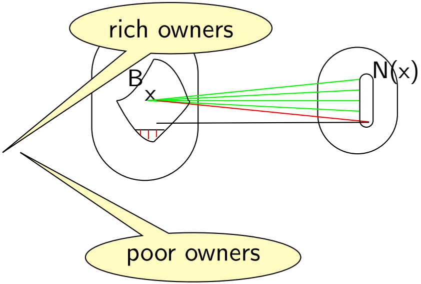

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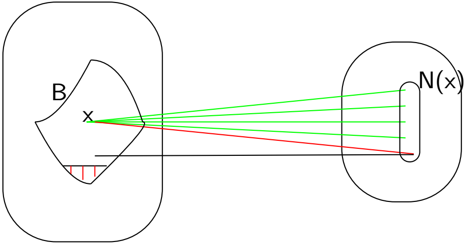
Theorem (based on the (Raz-Reingold-Vadhan 2002) extractor)

There exists a poly.-time computable (uniformly in n, k and $1/\delta$) graph with the rich owner property for parameters (k, δ) with:

- $L = \{0,1\}^n$
- $R = \{0,1\}^{k+O(\log^3(n/\delta))}$
- $D(\textit{left degree}) = 2^{O(\log^3(n/\delta))}$



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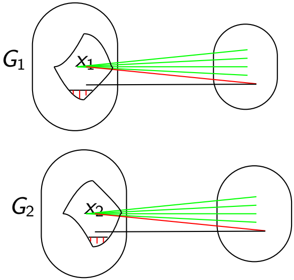

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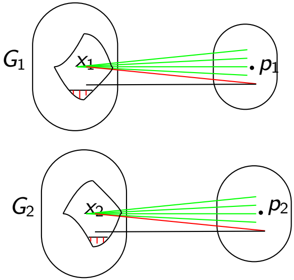

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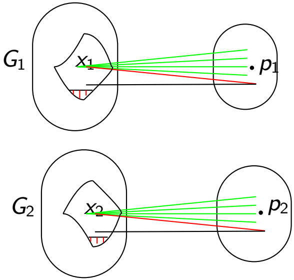

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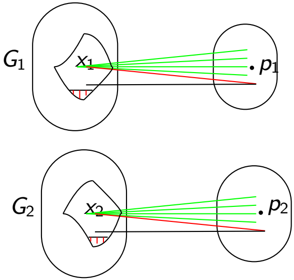
- **Idea:** For $i = 1, 2$, find B_i in the "small regime", containing x_i as a rich owner. Then with prob $1 - \delta$, x_i owns p_i, so from p_i we can reconstruct x_i.



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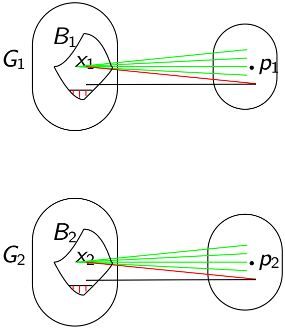

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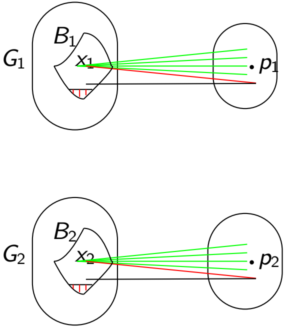

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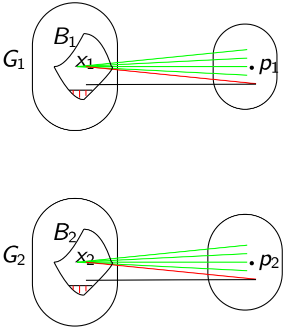

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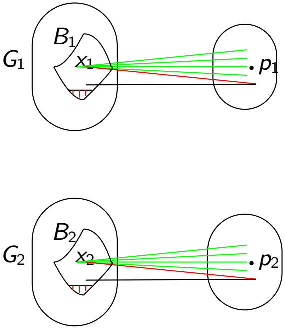

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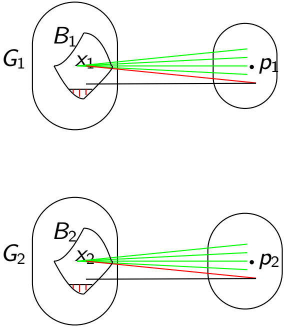

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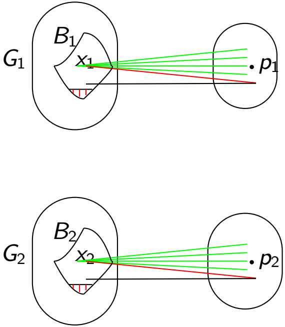

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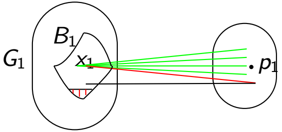

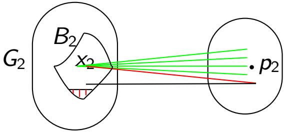

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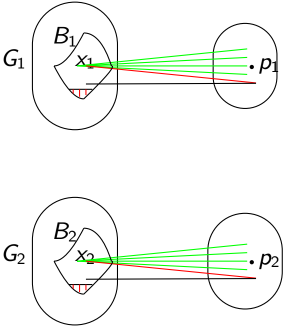

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- Using extra hashing, we can isolate x_1 and x_2 from the strings produced by the parallel procedures with incorrect guesses. Cost of hashing: $O(\log n)$ bits, because there are $O(n^3)$ parallel procedures.

Merci beaucoup.