Data Communication in the framework of Kolmogorov complexity

Marius Zimand (Towson University)

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Kolmogorov complexity

How complex is a string?
Kolmogorov complexity

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01101010 00001001 11100110 01100111 11110011 10111100 11001001 00001000

initial segment of $\sqrt{2} - 1$
Kolmogorov complexity

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01 repeated 32 times
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Kolmogorov complexity of a string = the length of a minimal description of the string.

Finding a minimal description of a string is a non-computable task.

Otherwise, we can compute for every $n$ the first string of length $n$ that has no description of length $n/2$. But this string can be described with $\log n$ bits.

"... in the framework of Kolmogorov complexity we have no compression algorithm and deal only with decompression algorithms."

We shall see natural circumstances where compression to close to minimum description length is not only effective but actually efficient (and decompression is effective but not efficient).
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We shall see natural circumstances where compression to close to minimum description length is not only effective but actually efficient (and decompression is effective but not efficient).
Warm-up puzzle

Alice and Bob want to agree on a secret key.
Problem is that we hear everything they say.
Alice knows line $L : y = a_1 x + a_0$;
Bob knows point $P: (b_1, b_2)$;
$L : 2n$ bits of information (intercept, slope in $\mathbb{F}_{2^n}$).
$P$: 2n bits of information (the 2 coord. in $\mathbb{F}_{2^n}$).
Total information in $(L, P) = 3n$ bits; mutual information of $L$ and $P = n$ bits.
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- $L: 2n$ bits of information (intercept, slope in $\mathbb{F}_{2^n}$).
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- Total information in $(L, P) = 3n$ bits; mutual information of $L$ and $P = n$ bits.

SOLUTION:

- Alice tells $a_1$ to Bob.
- Bob, knowing that $P \in L$, finds $L$.
- Alice and Bob use $a_0$ as a secret key.
- It works! We have heard $a_1$, but $a_1$ and $a_0$ are independent.
The real puzzle

Alice: \( x_1 \)
Bob: \( x_2 \)
Charlie: \( x_3 \)

Points \( x_1, x_2, x_3 \) belong to one line in the affine plane over \( \mathbb{F}_{2^n} \)

Each point has \( 2n \) points of information, but together they have \( 5n \) bits of information.
The real puzzle

Alice: $x_1$
Bob: $x_2$
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Points $x_1$, $x_2$, $x_3$ belong to one line in the affine plane over $\mathbb{F}_{2^n}$.

Each point has $2n$ points of information, but together they have $5n$ bits of information.

QUESTION: Can they agree on a secret key by discussing in this room, where we all hear what they say?
Kolmogorov complexity: measuring information in a string

\( C(x) := \text{length(shortest description of } x) \)
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**Formal Definition:**
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chose an algorithm \( \mathcal{A} \)
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**Formal Definition:**

chose an algorithm \( \mathcal{A} \)

\[ C_{\mathcal{A}}(x) := \min\{\text{length}(p) : \mathcal{A}(p) = x\} \]

\( p \) is called a program for \( x \) if \( \mathcal{A}(p) = x \).
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**Invariance Theorem:**
There exists an **optimal** \( \mathcal{U} \)
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such that \( C_{\mathcal{U}}(x) \leq C_{\mathcal{A}}(x) + O(1) \) for all other \( \mathcal{A} \).
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We fix **some** optimal \( \mathcal{U} \) once and forever.
Kolmogorov complexity: measuring information in a string

\[ C(x) := \text{size of a shortest program generating } x \]
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\[ x = 110111001\ldots101 \]

\[ n \text{ bits} \]

\[ x \text{ has a description of length } n + O(1). \]
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- \( C(x) \leq n + \text{const for all } x \) of length \( n \)
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\( x \) has a description of length \( n + O(1) \).

- \( C(x) \leq n + \text{const} \) for all \( x \) of length \( n \)
- \( C(x) \geq n - \text{const} \) for most \( x \) of length \( n \)
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\[ x = \underbrace{000000000\ldots000}_{n \text{ bits}} \]

\[ C(x) \leq \log n + O(1) \]
Information quantities for two strings $x, y$

- $C(x) := \text{length(shortest description of } x\text{)}$
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Other quantities: $C(y)$, $C(x, y)$
Information quantities for two strings $x, y$

- $C(x) := \text{length(shortest description of } x\text{)}$
  
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  Other quantities: $C(y), C(x, y)$

- $C(x \mid y) := \text{length(shortest description of } x \text{ given } y\text{)}$
  
  := size of a shortest program generating $x$ given $y$

  Another quantity: $C(y \mid x)$

- Mutual information of $x$ and $y$:
  
  $I(x : y) := C(x) - C(x \mid y)$
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- Mutual information of \( x \) and \( y \) :
  \[ I(x : y) := C(x) - C(x \mid y). \]

- **Chain Rule** [Kolmogorov, Levin]
  \[ C(x, y) =^+ C(x) + C(y \mid x) \]
  where the notation \( =^+ \) hides \( \pm O(\log n) \)

  **Corollary.**
  \[ I(x : y) =^+ C(x) + C(y) - C(x, y) =^+ I(y : x) \]
The word *random* is used in computer science in two ways:

(1) *random* process: a process whose outcome is uncertain, e.g. a series of coin tosses.

(2) *random* object: something that lacks regularities, patterns, is incompressible.

Information Theory (IT) focuses on (1).

Algorithmic Information Theory (AIT, also known as Kolmogorov complexity) focuses on (2).
IT vs. AIT

IT (à la Shannon)
- Data is the realization of a random variable $X$.
- The model: a stochastic process generates the data.
- Amount of information in the data: $H(X) = \sum p_i \log(1/p_i)$ (Shannon entropy).

AIT (Kolmogorov complexity)
- Data is just an individual string $x$.
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1/4, 1/4, 1/4, 1/4

101101000110010

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Short programs and communication protocols

Alice has $x$.

Bob has $y$.

They run an interactive protocol.

\[ t_1 \rightarrow t_2 \leftarrow t_3 \rightarrow \cdots \rightarrow t_k \rightarrow t_{k+1} \]

Bob has $x$.

QUESTION: What is the communication complexity?

Can it be $C(x \mid y)$? Is there a protocol that comes close to this?
Scenario: Alice and Bob are computationally unbounded

Alice has $x$, Bob has $y$. They run a protocol. At the end, Bob has $x$.

- If the protocol is **deterministic**, Alice needs to send $C(x)$ bits.
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- If the protocol is deterministic, Alice needs to send $C(x)$ bits.

(Buhrman, Koucky, Vereshchagin, 2014) There is a randomized protocol with communication complexity $C(x \mid y) + O(\sqrt{C(x \mid y)})$.

(Vereshchagin, 2014) The randomized communication complexity of computing $C(x \mid y)$ with precision $\epsilon/n$ is $0.99^n$. 

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- (Buhrman, Koucky, Vereshchagin, 2014) There is a **randomized** protocol with communication complexity \( C(x \mid y) + O(\sqrt{C(x \mid y)}) \).

- The difficult part: Alice needs to find \( C(x \mid y) \).
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- The difficult part: Alice needs to find $C(x \mid y)$.

- (Vereshchagin, 2014) The randomized communication complexity of computing $C(x \mid y)$ with precision $\epsilon n$ is $0.99n$. 
Scenario: Alice is algorithmically bounded

Alice has $x$, Bob has $y$. Alice wants a program for $x$ given $y$ (which she can send to Bob, to communicate $x$).

- A program $p$ for $x$ given $y$ is $c$-short, if $|p| \leq C(x) + c$. 

(Bauwens, Makhlin, Vereshchagin, Zimand, 2013) Alice can effectively compute on input $x$ a list with $O(n^2)$ elements that contains a $O(1)$-short program for $x$ given $y$. Such a list must have size $\Omega(n^2)$. (Teutsch, 2014) Alice can compute on input $x$ in polynomial time a list that contains a $O(1)$-short program for $x$ given $y$. (Zimand, 2014) Same as above, with list size $O(n^6 + \epsilon)$. 

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- (Zimand, 2014) Same as above, with list size $O(n^{6+\epsilon})$. 
Scenario: Alice is algorithmically bounded and holds advice information

Alice has \( x \), Bob has \( y \). Alice wants a program for \( x \) given \( y \) (which she can send to Bob, to communicate \( x \)).

- Assumption: Besides \( x \), Alice has some information about \( x \) and \( y \) (called advice).
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- **Muchnik’s theorem, 2001**: Alice on input $x$ and some $O(\log n)$-long advice can compute a 0-short program for $x$ given $y$. 
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- (Musatov, Romashchenko, Shen, 2009) **Space-bounded version of Muchnik’s Th.**:

  For every space bound $s$, Alice on $x$ and some $O(\log^3 n)$-long advice can compute in polynomial space a program $p$ for $x$ given $y$ with space complexity $O(s) + \text{poly}(n)$ and $|p| = C^{\text{space}=s}(x \mid y) + O(\log n)$. 

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Scenario: Alice is algorithmically bounded and knows $C(x \mid y)$.

Alice has $x$, Bob has $y$. Alice wants a program for $x$ given $y$ (which she can send to Bob, to communicate $x$).

- Assumption: Besides $x$, Alice has some information about $x$ and $y$, $C(x \mid y)$.
Scenario: Alice is algorithmically bounded and knows $C(x \mid y)$.

Alice has $x$, Bob has $y$. Alice wants a program for $x$ given $y$ (which she can send to Bob, to communicate $x$).

- Assumption: Besides $x$, Alice has some information about $x$ and $y$ $C(x \mid y)$.
- Consider the special case $y = \text{empty string}$. 
Scenario: Alice is algorithmically bounded and knows $C(x \mid y)$.

Alice has $x$, Bob has $y$. Alice wants a program for $x$ given $y$ (which she can send to Bob, to communicate $x$).

- Assumption: Besides $x$, Alice has some information about $x$ and $y$ $C(x\mid y)$.
- Consider the special case $y = \text{empty string}$.
- Alice on input $x$ and $C(x)$ can find a 0-short program for $x$ by exhaustive search, but this is VERY SLOW.
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- Alice on input $x$ and $C(x)$ can find a 0-short program for $x$ by exhaustive search, but this is VERY SLOW.
- (Bauwens, Zimand, 2014) Let $t$ be any computable function. If an algorithm on input $(x, C(x))$ finds a program $p$ for $x$ in time $t(n)$, then for infinitely many $x$, $|p| = C(x) + \Omega(n)$. 

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Scenario: Alice is algorithmically bounded and knows $C(x | y)$.

Alice has $x$, Bob has $y$. Alice wants a program for $x$ given $y$ (which she can send to Bob, to communicate $x$).

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- (Bauwens, Zimand, 2014) Alice on input $(x, C(x))$ can compute in probabilistic polynomial time a $O(\log^2(n/\epsilon))$-short program for $x$ given $y$, with probability error $\epsilon$. 
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- If we drop the poly time requirement, the overhead can be reduced to $O(\log n)$.
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- (Bauwens, Zimand, 2014) Alice on input $(x, C(x))$ can compute in probabilistic polynomial time a $O(\log^2(n/\epsilon))$-short program for $x$ given $y$, with probability error $\epsilon$.
- If we drop the poly time requirement, the overhead can be reduced to $O(\log n)$.
- The overhead cannot be less than $\log n - \log \log n - O(1)$, for total computable compressors.
Scenario: Alice is algorithmically bounded and knows an upper bound of $C(x \mid y)$.

Alice has $x$, Bob has $y$. Alice wants a program for $x$ given $y$ (which she can send to Bob, to communicate $x$).

- Assumption: Besides $x$, Alice has $\mathcal{C}(x \mid y)$, $m \geq C(x \mid y)$
Scenario: Alice is algorithmically bounded and knows an upper bound of $C(x \mid y)$.

Alice has $x$, Bob has $y$. Alice wants a program for $x$ given $y$ (which she can send to Bob, to communicate $x$).

- Assumption: Besides $x$, Alice has $C(x \mid y) \quad m \geq C(x \mid y)$
- Consider the special case $y = \text{empty string}$.
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Alice has $x$, Bob has $y$. Alice wants a program for $x$ given $y$ (which she can send to Bob, to communicate $x$).

- Assumption: Besides $x$, Alice has $C(x \mid y) \geq C(x \mid y)$
- Consider the special case $y = \text{empty string}$.
- Alice on input $x$ and $k$ can find a program $p$ for $x$ with $|p| \leq m$ by exhaustive search, but this is VERY SLOW.
Scenario: Alice is algorithmically bounded and knows an upper bound of $C(x \mid y)$.

Alice has $x$, Bob has $y$. Alice wants a program for $x$ given $y$ (which she can send to Bob, to communicate $x$).

- Assumption: Besides $x$, Alice has $C(x \mid y) \geq C(x \mid y)$
- Consider the special case $y = \text{empty string}$.
- Alice on input $x$ and $k$ can find a program $p$ for $x$ with $|p| \leq m$ by exhaustive search, but this is VERY SLOW.
- (Zimand, 2017) (Bauwens, Zimand, 2019) Alice on input $(x, m)$ can compute in probabilistic polynomial time a program for $x$ given $y$ of length $m + O(\log^2(n/\epsilon))$, with probability error $\epsilon$. 
Theorem (Bauwens, Zimand, 2019)

(Bauwens, Zimand, 2019) Alice on input $(x, m)$, where $m \geq C(x \mid y)$, can compute in probabilistic polynomial time a program for $x$ given $y$ of length $m + O(\log^2(n/\epsilon))$, with probability error $\epsilon$.

- $f : L \times [D] \rightarrow R$, used for fingerprinting.
- $f(x, 1), \ldots, f(x, D)$ are the fingerprints of $x$.
- $X$ is the list of candidates, we want to identify which candidate is $x$.
- A fingerprint is heavy for $X$, if it has more $2D$ pre-images in $X$.
- $x$ is $\epsilon$-defective for $X$ if it has more than $\epsilon D$ heavy fingerprints.
Theorem (Bauwens, Zimand, 2019)

(Bauwens, Zimand, 2019) Alice on input \((x, m)\), where \(m \geq C(x \mid y)\), can compute in probabilistic polynomial time a program for \(x\) given \(y\) of length \(m + O(\log^2 (n/\epsilon))\), with probability error \(\epsilon\).

\[
L = \{0,1\}^n
\]

\[
R = \{0,1\}^m
\]

- \(f: \{0,1\}^n \times [D] \rightarrow \{0,1\}^m\) is a \(k \rightarrow \epsilon\) \(k\)-condenser, if for every r.v. \(X\) with min-entropy \(k\), \(f(X,U_D)\) is \(\epsilon\)-close to having min-entropy \(k\).
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**Pr \([X = x] \leq 2^{-k}\) for all \(x\)**

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Buhrman, Fortnow, Laplante 2001 used extractors for a related problem

\[ f : \{0, 1\}^n \times [D] \rightarrow \{0, 1\}^m \text{ is a } k \rightarrow \epsilon \text{ \(k\) condenser}, \text{ if for every r.v. } X \text{ with min-entropy } k, f(X, U_D) \text{ is } \epsilon\text{-close to having min-entropy } k. \]
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- \(f : \{0, 1\}^n \times [D] \rightarrow \{0, 1\}^m\) is an \(\epsilon\) conductor, if it is a \(k \rightarrow \epsilon k\) condenser for every \(k \leq m\).
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- If \(f : \{0, 1\}^n \times [D] \rightarrow \{0, 1\}^m\) is an \(\epsilon\) conductor, for every \(X\), the fraction of \(4\epsilon\)-defective elements in \(X\) is at most \(1/2\).
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- (Bauwens, Zimand 2019) There exists poly-time \(\epsilon\) conductor with \(D = 2^{\log^2(n/\epsilon)}\), for every \(m \leq n\).
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- $f : \{0, 1\}^n \times [D] \to \{0, 1\}^m$ is a $k \to \epsilon$ $k$ condenser, if for every r.v. $X$ with min-entropy $k$, $f(X, U_D)$ is $\epsilon$-close to having min-entropy $k$.

- $f : \{0, 1\}^n \times [D] \to \{0, 1\}^m$ is an $\epsilon$ conductor, if it is a $k \to \epsilon$ condenser for every $k \leq m$.

- If $f : \{0, 1\}^n \times [D] \to \{0, 1\}^m$ is an $\epsilon$ conductor, for every $X$, the fraction of $4\epsilon$-defective elements in $X$ is at most $1/2$.

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building on the Guruswami-Umans-Vadhan extractor
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(Bauwens, Zimand, 2019) Alice on input \((x, m)\), where \(m \geq C(x \mid y)\), can compute in probabilistic polynomial time a program for \(x\) given \(y\) of length \(m + O(\log^2(n/\epsilon))\), with probability error \(\epsilon\).

- \(x\) has complexity \(C(x) \leq m\).
- \(f : \{0, 1\}^n \times [D] \rightarrow \{0, 1\}^m\), the explicit \(\epsilon\)-conductor.
- \(x\) has complexity \(C(x) \leq m\).
- \(f(x, 1), \ldots, f(x, D)\) are the fingerprints of \(x\).

Compress \(x\): pick randomly \(p\), one of the fingerprints. Append \(h\), a short hash-code of \(x\). Output \((p, h)\). Length: \(m + |h|\).

Decompression: we want to reconstruct \(x\) from \((p, h)\).

\(X\) the set of strings with complexity \(\leq m\) (list of candidates). we want to identify which candidate is \(x\).
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probability error \(\epsilon\).

- Decompression: we want to reconstruct \(x\) from \((p, h)\).
- \(X\) the set of strings with complexity \(\leq m\) (list 
of candidates). We want to identify which candidate is \(x\).
- Case 1: \(x\) is non-defective. With prob \(1 - \epsilon\), we 
reduce the list of candidates to the 
\(2D\)-preimages of \(p\).
- Case 2: \(x\) is defective. We reduce the list of 
candidates to the set of defective elements, so 
we reduce the list by \(1/2\).
- Continue recursively with fewer candidates.
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- Problem: We do not know which of Case 1 or Case 2 is true.
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- Case 1: \(x\) is non-defective. With prob \(1 - \epsilon\), we reduce the list of candidates to the 2D-preimages of \(p\).
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**Problem:** We do not know which of Case 1 or Case 2 is true.

- We collect the candidates as if Case 1 is true, so we keep only the first 2D preimages of \(p\). Then reduce as in Case 2.
- At the end we have collected \(m \times 2D\) candidates.
- We identify \(x\) using \(h\), the short hash code.
Distributed compression: a simple example

- Alice knows a line $\ell$; Bob knows a point $P \in \ell$; They want to send $\ell$ and $P$ to Zack.
- $\ell : 2n$ bits of information (intercept, slope in GF[$2^n$]).
- $P : 2n$ bits of information (the 2 coord. in GF[$2^n$]).
- Total information in $(\ell, P) = 3n$ bits; mutual information of $\ell$ and $P = n$ bits.
- If Alice and Bob get together, they need to send $3n$ bits. What if they compress separately?
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**QUESTION 1:**

Alice can send $2n$ bits, and Bob $n$ bits. Is the geometric correlation between $\ell$ and $P$ crucial for these compression lengths?

**Ans:** No. Same is true (modulo a polylog$(n)$ overhead.) if Alice and Bob each have $2n$ bits of information, with mutual information $n$, in the sense of Kolmogorov complexity.
Distributed compression: a simple example

- Alice knows a line \( \ell \); Bob knows a point \( P \in \ell \); They want to send \( \ell \) and \( P \) to Zack.
- \( \ell \): \( 2n \) bits of information (intercept, slope in \( \text{GF}[2^n] \)).
- \( P \): \( 2n \) bits of information (the 2 coord. in \( \text{GF}[2^n] \)).
- Total information in \( (\ell, P) = 3n \) bits; mutual information of \( \ell \) and \( P = n \) bits.
- If Alice and Bob get together, they need to send \( 3n \) bits. What if they compress separately?

QUESTION 2:

Can Alice send 1.5\( n \) bits, and Bob 1.5\( n \) bits? Can Alice send 1.74\( n \) bits, and Bob 1.26\( n \) bits?

Ans: Yes and Yes (modulo a polylog\((n)\) overhead.)
Distributed compression (IT view): Slepian-Wolf Theorem

- The classic Slepian-Wolf Th. is the analog of Shannon Source Coding Th. for the distributed compression of memoryless sources.
- Memoryless source: \((X_1, X_2)\) consists of \(n\) independent draws from a joint distribution \(p(b_1, b_2)\) on pair of bits.
- Encoding: \(E_1 : \{0, 1\}^n \rightarrow \{0, 1\}^{n_1}, E_2 : \{0, 1\}^n \rightarrow \{0, 1\}^{n_2}\).
- Decoding: \(D : \{0, 1\}^{n_1} \times \{0, 1\}^{n_2} \rightarrow \{0, 1\}^n \times \{0, 1\}^n\).
- Goal: \(D(E_1(X_1), E_2(X_2)) = (X_1, X_2)\) with probability \(1 - \epsilon\).

It is necessary that \(n_1 + n_2 \geq H(X_1, X_2) - \epsilon n\), \(n_1 \geq H(X_1 | X_2) - \epsilon n\), \(n_2 \geq H(X_2 | X_1) - \epsilon n\).

Theorem (Slepian, Wolf, 1973)

There exist encoding/decoding functions \(E_1, E_2\) and \(D\) satisfying the goal for all \(n_1, n_2\) satisfying

\[n_1 + n_2 \geq H(X_1, X_2) + \epsilon n, \quad n_1 \geq H(X_1 | X_2) + \epsilon n, \quad n_2 \geq H(X_2 | X_1) + \epsilon n.\]

It holds for any constant number of sources.
Slepian-Wolf Th.: Some comments

**Theorem (Slepian, Wolf, 1973)**

There exist encoding/decoding functions $E_1$, $E_2$ and $D$ such that

\[ n_1 + n_2 \geq H(X_1, X_2) + \epsilon_n, \]
\[ n_1 \geq H(X_1 \mid X_2) + \epsilon_n, \]
\[ n_2 \geq H(X_2 \mid X_1) + \epsilon_n. \]

- Even if $(X_1, X_2)$ are compressed together, the sender still needs to send $\approx H(X_1, X_2)$ many bits.
- **Strength of S.-W. Th.**: distributed compression = centralized compression, for memoryless sources.
- **Shortcoming of S.-W. Th.**: Memoryless sources are very simple. The theorem has been extended to stationary and ergodic sources (Cover, 1975), which are still pretty lame.
Recall: Alice knows a line $\ell$; Bob knows a point $P \in \ell$; They want to send $\ell$ and $P$ to Zack.

There is no generative model.

Correlation can be described with the complexity profile: $C(\ell) = 2n$, $C(P) = 2n$, $C(\ell, P) = 3n$.

Is it possible to have distributed compression based only on the complexity profile?

If yes, what are the possible compression lengths?
Recall: Alice knows a line $\ell$; Bob knows a point $P \in \ell$; They want to send $\ell$ and $P$ to Zack.

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Is it possible to have distributed compression based only on the complexity profile?

If yes, what are the possible compression lengths?

**Necessary conditions:** Suppose we want encoding/decoding procedures so that $D(E_1(x_1), E_2(x_2)) = (x_1, x_2)$ with probability $1 - \epsilon$, for all strings $x_1, x_2$.

Then, for infinitely many $x_1, x_2$,

\[
|E_1(x_1)| + |E_2(x_2)| \geq C(x_1, x_2) + \log(1 - \epsilon) - O(1)
\]

\[
|E_1(x_1)| \geq C(x_1 | x_2) + \log(1 - \epsilon) - O(1)
\]

\[
|E_2(x_2)| \geq C(x_2 | x_1) + \log(1 - \epsilon) - O(1)
\]
Kolmogorov complexity version of the Slepian-Wolf Theorem

Theorem ((Z. 2017), (Bauwens, Z. 2019))

There exist probabilistic poly.-time algorithms $E$ and algorithm $D$ such that for all integers $n_1, n_2$ and $n$-bit strings $x_1, x_2$,

if $n_1 + n_2 \geq C(x_1, x_2)$, $n_1 \geq C(x_1 \mid x_2)$, $n_2 \geq C(x_2 \mid x_1)$,

then

- $E$ on input $(x_i, n_i)$ outputs a string $p_i$ of length $n_i + O(\log^2 n)$, for $i = 1, 2$,
- $D$ on input $(p_1, p_2)$ outputs $(x_1, x_2)$ with probability 0.99.

There is an analogous version for any constant number of sources.
Alice has $x_1$ and $n_1$. 

Bob has $x_2$ and $n_2$. $n_1, n_2$ satisfy the Slepian-Wolf constraints:

$$n_1 + n_2 \geq C(x_1, x_2),$$
$$n_1 \geq C(x_1 | x_2),$$
$$n_2 \geq C(x_2 | x_1).$$

Alice uses a conductor with output size $n_1$. Bob uses a conductor with output size $n_2$.

Alice compresses $x_1$ by choosing a random neighbor $p_1$ + short hash-code $h_1$.

Bob compresses $x_2$ by choosing a random neighbor $p_2$ + short hash-code $h_2$. 

$L = \{0, 1\}^n$ $R = \{0, 1\}^n_1$ $D = \{0, 1\}^n$ $M = \{0, 1\}^n_2$
Proof sketch (1/2)

- Alice has $x_1$ and $n_1$.
- Bob has $x_2$ and $n_2$. 
Proof sketch (1/2)

- Alice has $x_1$ and $n_1$.
- Bob has $x_2$ and $n_2$.
- $n_1$, $n_2$ satisfy the Slepian-Wolf constraints:
  
  $$n_1 + n_2 \geq C(x_1, x_2), \quad n_1 \geq C(x_1 \mid x_2), \quad n_2 \geq C(x_2 \geq x_1).$$


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  \[ n_1 + n_2 \geq C(x_1, x_2), \quad n_1 \geq C(x_1 \mid x_2), \quad n_2 \geq C(x_2 \geq x_1). \]
- Alice uses a conductor with output size $= n_1$.
- Bob uses a conductor with output size $= n_2$. 
Alice has \( x_1 \) and \( n_1 \).

Bob has \( x_2 \) and \( n_2 \).

\( n_1, n_2 \) satisfy the Slepian-Wolf constraints:

\[
\begin{align*}
    n_1 + n_2 & \geq C(x_1, x_2), \\
    n_1 & \geq C(x_1 \mid x_2), \\
    n_2 & \geq C(x_2 \geq x_1).
\end{align*}
\]

Alice uses a conductor with output size = \( n_1 \).

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Alice compresses \( x_1 \) by choosing a random neighbor \( p_1 + \) short hash-code \( h_1 \).
Proof sketch (1/2)

- Alice has $x_1$ and $n_1$.
- Bob has $x_2$ and $n_2$.
- $n_1, n_2$ satisfy the Slepian-Wolf constraints:
  \[ n_1 + n_2 \geq C(x_1, x_2), n_1 \geq C(x_1 \mid x_2), n_2 \geq C(x_2 \geq x_1). \]
- Alice uses a conductor with output size $= n_1$.
- Bob uses a conductor with output size $= n_2$.
- Alice compresses $x_1$ by choosing a random neighbor $p_1 + \text{short hash-code } h_1$.
- Bob compresses $x_2$ by choosing a random neighbor $p_2 + \text{short hash-code } h_2$. 
Proof-sketch (2/2)

- How to reconstruct \((x_1, x_2)\) from \((p_1, h_1)\) and \((p_2, h_2)\)
- Enumerate the initial list of candidates: all pairs \(x'_1, x'_2\) with
  \[n_1 + n_2 \geq C(x'_1, x'_2), n_1 \geq C(x'_1 \mid x'_2), n_2 \geq C(x'_2 \geq x'_1)\].
- Apply a cascade of two filters to each enumerated pair.
- Pair \((x'_1, \ast)\) passes the first filter if \((p_1, h_1)\) is the compressed code of \(x'_1\).
- Pair \((\ast, x'_2)\) passes the second filter if \((p_2, h_2)\) is the compressed code of \(x'_2\).
- With high probability, only \((x_1, x_2)\) survive the two filters.
Some comments

- Compression takes polynomial time. Decompression is slower than any computable function. This is unavoidable at this level of optimality (compression at close to minimum description length).
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- Compression for individual strings is also done by Lempel-Ziv algorithms. They compress optimally for finite-state procedures. We compress at close to minimum description length.
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- The classical S.-W. Th. can be obtained from the Kolmogorov complexity version (because if $X$ is memoryless, $H(X) - c\epsilon \sqrt{n} \leq C(X) \leq H(X) + c\epsilon \sqrt{n}$ with prob. $1 - \epsilon$).
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- The $O(\log^2 n)$ overhead can be reduced to $O(\log n)$, but compression is no longer in polynomial time.
Operational characterization of mutual information

\[ C(x) = \text{length of a shortest description of } x. \]
\[ C(x \mid y) = \text{length of a shortest description of } x \text{ given } y. \]

Mutual information of \( x \) and \( y \) is defined by a formula:
\[ I(x : y) = C(x) + C(y) - C(x, y). \]
Also, \[ I(x : y) = C(x) - C(x \mid y), \]
\[ I(x : y) = C(y) - C(y \mid x) \]
\[ \left( =^+ \text{ hides } \pm O(\log n) \right) \]

All the regions except the center have an operational meaning.
Operational characterization of mutual information

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Also, 

$$I(x : y) =^+ C(x) - C(x | y),$$

$$I(x : y) =^+ C(y) - C(y | x)$$

(= hides $\pm O(\log n)$)

All the regions except the center have an operational meaning.

Does $I(x : y)$ have an operational meaning?
Question: Can mutual information be “materialized”? 
Mutual information and secret key agreement

- Question: Can mutual information be “materialized”? 
- Answer: YES.
Question: Can mutual information be “materialized”?  
Answer: YES.

Mutual information of strings $x, y =$ length of the longest shared secret key that Alice having $x$ and Bob having $y$ can establish via a randomized protocol.
Question: Can mutual information be “materialized”?
Answer: YES.

Mutual information of strings $x, y = \text{length of the longest shared secret key that Alice having } x \text{ and Bob having } y \text{ can establish via a randomized protocol.}$

This was known in the setting of Information Theory (Shannon entropy, etc.) for memoryless and stationary ergodic sources.

(Romashchenko, Z., 2018) Characterization holds in the framework of Kolmogorov complexity.
Secret key agreement protocol:

- Alice knows $x$.
- Bob knows $y$.
- They exchange messages and compute a shared secret key $z$.
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(1) Alice and Bob also know how their $x$ and $y$ are correlated.
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Our setting:

1. Alice and Bob also know how their $x$ and $y$ are correlated. Technically, they know the complexity profile of $x$ and $y$: $(C(x), C(y), C(x, y))$.
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Theorem (Characterization of the mutual information)

1. There is a protocol that for every $n$-bit strings $x$ and $y$ allows to compute with high probability a shared secret key of length $I(x : y)$ (up to $-\Theta(\log n)$).
2. No protocol can produce a longer shared secret key (up to $+\Theta(\log n)$).
Characterization of mutual information: the positive part

Theorem

There exists a secret key agreement protocol with the following property: if

- Alice knows $x$, $\epsilon$, and the complexity profile of $(x, y)$,
- Bob knows $y$, $\epsilon$, and the complexity profile of $(x, y)$,

then with probability $1 - \epsilon$ they obtain a string $z$ such that,

$$|z| \geq I(x : y) - O(\log(n/\epsilon))$$

and $C(z | \text{transcript}) \geq |z| - O(\log(1/\epsilon))$. 
Theorem

There exists a secret key agreement protocol with the following property: if

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then with probability $1 - \epsilon$ they obtain a string $z$ such that,

$$|z| \geq I(x : y) - O(\log(n/\epsilon))$$  /* common key of size $\geq^+ I(x : y)$ */

and $C(z \mid \text{transcript}) \geq |z| - O(\log(1/\epsilon))$. /* no information leakage */
Secret key agreement: sketch of the general protocol

- Alice and Bob want to agree on a secret key.
- they can only communicate through a public channel.
- Alice knows $x$; Bob knows $y$;
- $C(x \mid y) = n_1$
- $C(y \mid x) = n_2$
- $I(x : y) = n_0$. 

Protocol:
- Alice sends to Bob a program $p$ of $x$ given $y$ of size $= n_1$.
- Bob (knowing $y$) reconstructs $x$.
- Alice and Bob compute (independently) a program $z$ of $x$ given $p$ of size $= n_0$.
- Adversary gets $p$ but learns nothing about $z$. 

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- Adversary gets $p$ but learns nothing about $z$. 
Characterization of mutual information: the negative part.

**Theorem**

Let $x$ and $y$ be input strings of length $n$ on which the protocol succeeds with error probability $\epsilon$ so that with prob $1 - \epsilon$ Alice and Bob have at the end the same $z$, and $C(z \mid t) \geq |z| - \delta(n)$.

Then with probability $\geq 1 - O(\epsilon)$ we have

$|z| \leq I(x : y) + \delta(n) + O(\log(n/\epsilon))$. 
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Conditional information inequality

- simple part: if no communication, then key $\leq I(x : y)$
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Under the hood:

**Conditional information inequality**

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- simple part: if no communication, then $\textbf{key} \leq I(x : y)$
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Characterization of mutual information: the negative part.

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- hard part: $I(x : y \mid \text{transcript}) \leq I(x : y)$
- technical lemma: $C(\text{transcript} \mid x) + C(\text{transcript} \mid y) \leq C(\text{transcript})$
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Fact: Our protocol for secret key agreement produces a key of length $\approx I(x : y)$ and has communication complexity $\approx \min\{C(x | y), C(y | x)\}$. 
Communication complexity for secret key agreement

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**Theorem (Somewhat informal)**

*If the communication complexity of a protocol with public randomness is $< 0.999 \cdot \min\{C(x | y), C(y | x)\}$, then the size of the obtained common secret key is $\ll 0.001 \cdot I(x : y)$.***

Under the hood: common information is far less than mutual information (Gács & Körner 1970s; Kolmogorov seminar in 1990s; Muchik & A.R. 2000s)

Opposition stochastic/nonstochastic objects (Shen 1983; Razenshteyn 2011)

Open question: What is the communication complexity for the model with private random bits?
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- a substitution for the Diffie–Hellman protocol in post-quantum cryptography
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- fun: “operational” characterization of the mutual information, answering an old folklore question
- real fun: many interesting techniques in the proofs
- a gadget (“primitive”) for more complex crypto protocols
  possible area of applications: biometrics
Previous works

History

Similar results were known in Shannon's information theory:
Ahlswede and Csiszár [1993] and Maurer [1993]: the optimal size of the common secret key for two parties
Csiszár and Narayan [2004]: the optimal size of the common secret key for \( \ell > 2 \) parties
Tyagi [2013]: communication complexity of the protocols

Formal difference previous works:
- random variables & Shannon's entropy

Our work:
- binary strings & Kolmogorov complexity

1st substantial difference
Previous works: \((X, Y)\) from random memoryless / stationary ergodic sources
Our work: no specific structure on \(X\) and \(Y\)

2nd substantial difference
Previous works: protocols work for most admissible pairs \((X, Y)\)
Our work: protocols work for all admissible pairs \((X, Y)\)
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First substantial difference:

- Previous works: random variables and Shannon’s entropy.
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Second substantial difference:

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The puzzle

Alice: \( x_1 \)
Bob: \( x_2 \)
Charlie: \( x_3 \)

points \( x_1, x_2, x_3 \) belong to one line in the affine plane over \( \mathbb{F}_{2^n} \)

Each point has \( 2n \) points of information, but together they have \( 5n \) bits of information.
The puzzle

Alice: $x_1$
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points $x_1, x_2, x_3$ belong to one line in the affine plane over $\mathbb{F}_{2^n}$

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QUESTION: Can they agree on a secret key by discussing in this room, where we all hear what they say?
The puzzle: Solution

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Alice, Bob, Charlie send, respectively, $n_1$, $n_2$, $n_3$ bits, so that at the end they all know all points.
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- Alice, Bob, Charlie send, respectively, \(n_1, n_2, n_3\) bits, so that at the end they all know all points.
- Information requirements: \(n_i \geq n, n_i + n_j \geq 3n\), for all \(i, j \in \{1, 2, 3\}\).
The puzzle: Solution

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- \( n_1, n_2, n_3 = 1.5n \) satisfy the requirements. By S.-W. Th., they can each send \( 1.5n \) bits.
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- We have heard 4.5\( n \) bits, but they have 5\( n \) bits.
Alice, Bob, Charlie send, respectively, $n_1, n_2, n_3$ bits, so that at the end they all know all points.

Information requirements: $n_i \geq n$, $n_i + n_j \geq 3n$, for all $i, j \in \{1, 2, 3\}$.

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We have heard $4.5n$ bits, but they have $5n$ bits.

They compress their $5n$ bits conditional to our $4.5n$ bits, and obtain $0.5n$ secret bits.
The puzzle: Solution

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- They compress their $5n$ bits conditional to our $4.5n$ bits, and obtain $0.5n$ secret bits.
- (Romashchenko, Z. 2018) This is the best they can do, they cannot obtain a longer secret key.
Thank you

References:

